

3-Dimensional Space and Sketches

Notes, Examples, and Exercises (with solutions)

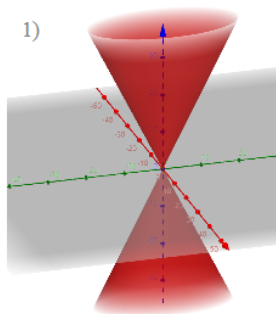
Topics include cylindrical and spherical coordinates, vectors, quadric surfaces, conics, rotation, coordinate system

Describe the equations:

3-dimension coordinate system: quadric surfaces

$$1) 16x^2 + 4y^2 - z^2 = 0$$

2 hyperbolas and an ellipse
"cones sprouting from the origin"



Note the "xy-component" (found by eliminating/making z constant)

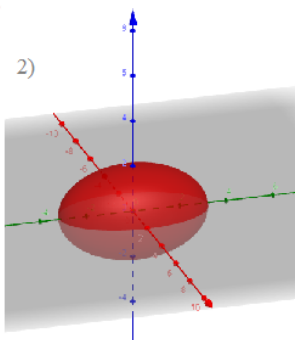
"xz-component" (found by eliminating/making y constant)

"yz-component" (found by eliminating/making x constant)

$$2) 5x^2 + 2y^2 + 5z^2 = 20$$

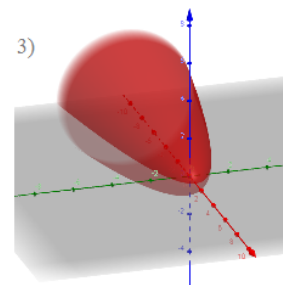
The components
include 2 ellipses
and a circle

"ellipsoid" (or, spherical ellipsoid)



$$3) x + y^2 + z^2 = 1$$

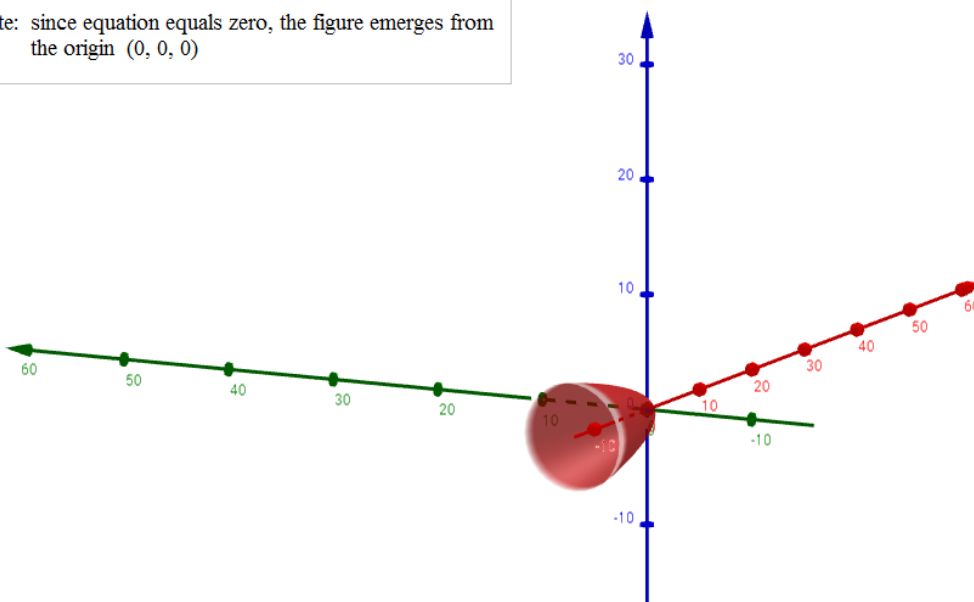
2 parabolas and a circle
("bowl shaped")



Example: $8x + 5y^2 + 5z^2 = 0$

components are parabola (when z is constant)
parabola (when y is constant)
circle (when x is constant)

Note: since equation equals zero, the figure emerges from the origin (0, 0, 0)



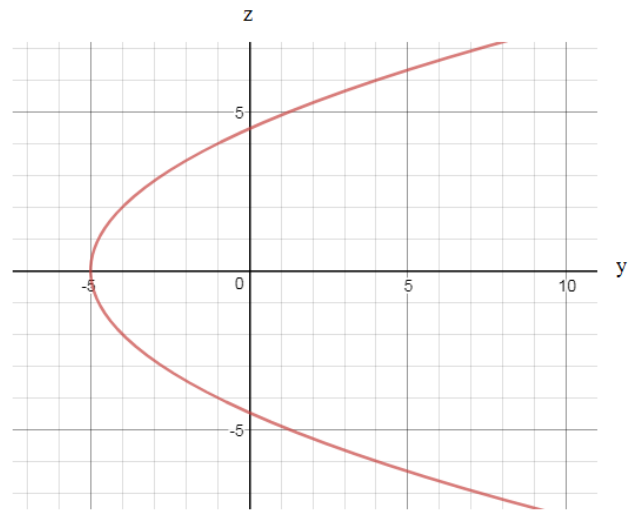
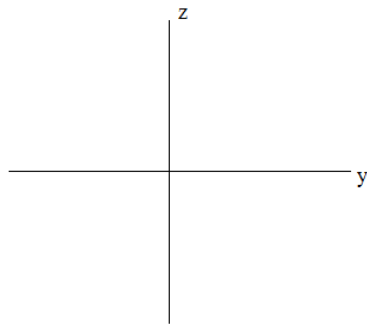
Example: Sketch $z^2 - 4y - 20 = 0$ revolved around the z-axis

Sketching 3-Dimensional Conic

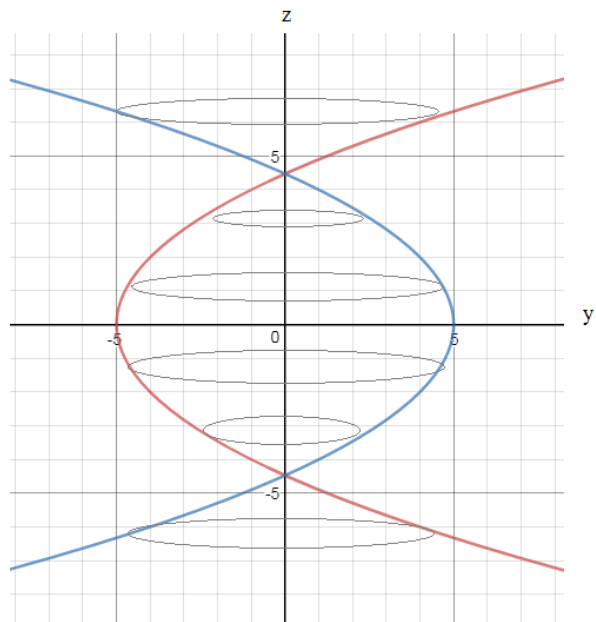
Sketch the curve in the 2-dimensional plane yz

$$z^2 - 20 = 4y$$

$$y = \frac{1}{4}z^2 - 5$$



Then, revolve around the z-axis...



another method: substitute $\sqrt{x^2 + y^2}$ in place of y...


 this represents the radius
 of each circle of rotation
 (i.e. each partition or cross section)

$$z^2 - 4y - 20 = 0$$

$$z^2 + 4\sqrt{x^2 + y^2} = 20$$

$$4\sqrt{x^2 + y^2} = 20 - z^2$$

$$16x^2 + 16y^2 = 400 - 40z^2 + z^4$$

$$16x^2 + 16y^2 = (20 - z^2)^2$$

Example: Change $(-1, 1, -2)$ into cylindrical coordinates... Then, into spherical coordinates....

Cylindrical coordinates are (r, Θ, z)

To find r and Θ , use the same approach as polar coordinates...

$$x = r \cos \Theta$$

$$y = r \sin \Theta$$

$$r^2 = x^2 + y^2$$

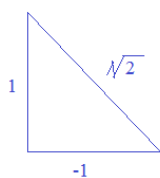
$$r^2 = (-1)^2 + (1)^2$$

$$r = \sqrt{2}$$

$$\text{Then, } -1 = \sqrt{2} \cos \Theta$$

$$1 = \sqrt{2} \sin \Theta$$

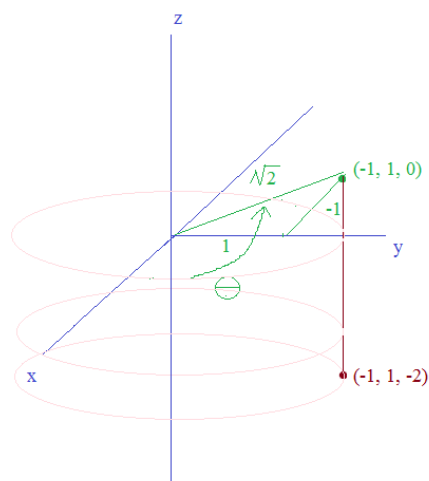
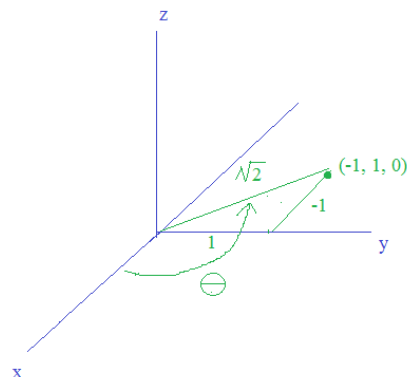
since $\sin \Theta$ is positive, and $\cos \Theta$ is negative, the angle falls in quadrant II



$$\Theta = 135 \text{ degrees... } \frac{3\pi}{4}$$

Then, the z coordinate is $z = -2$

$$\left(\sqrt{2}, \frac{3\pi}{4}, -2 \right)$$



Spherical coordinates are (p, Θ, Φ)

Θ is the "swivel around the xy -plane"

Φ is the angle between r and p

p is the distance between the origin and (x, y, z)

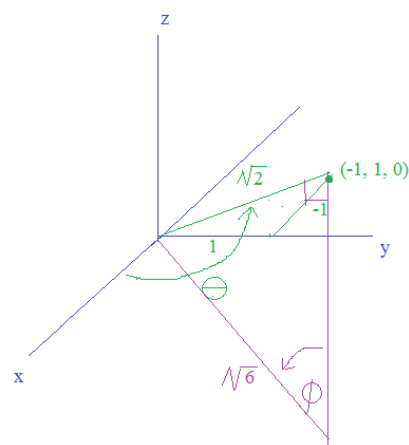
$$p^2 = (-1)^2 + (1)^2 + (-2)^2$$

$$p = \sqrt{6}$$

from above, we know $\Theta = \frac{3\pi}{4}$

$$\text{and, } r = \sqrt{2} \quad \sin \Phi = \frac{r}{p} = \frac{\sqrt{2}}{\sqrt{6}}$$

$$\left(\sqrt{6}, \frac{3\pi}{4}, \sin^{-1} \frac{1}{\sqrt{3}} \right)$$



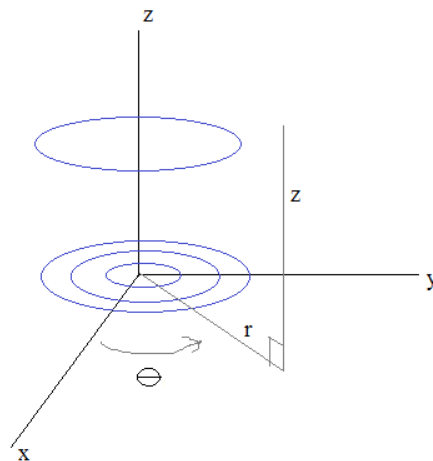
Cylindrical Coordinate System

$$(r, \Theta, z)$$

Converting rectangular coordinates (x, y, z) to cylindrical coordinates (r, Θ, z)

$$\begin{aligned} r &= \sqrt{x^2 + y^2} & x^2 + y^2 &= r^2 \\ \Theta &= \tan^{-1}\left(\frac{y}{x}\right) & x &= r \cos \Theta \\ z &= z & y &= r \sin \Theta \end{aligned}$$

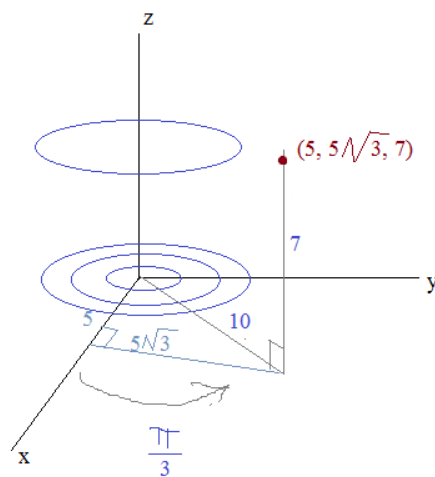
It's polar coordinate system with an added dimension!



Example: Convert $(5, 5\sqrt{3}, 7)$ to cylindrical coordinates.

$$\begin{aligned} z &= 7 \\ r &= \sqrt{5^2 + (5\sqrt{3})^2} = 10 \\ \tan \Theta &= \frac{5\sqrt{3}}{5} \\ \Theta &= \tan^{-1} \sqrt{3} = 60^\circ \text{ or } \frac{\pi}{3} \\ &(10, \frac{\pi}{3}, 7) \end{aligned}$$

$$\begin{aligned} \text{quick check: } x &= r \cos \Theta = 10 \cos\left(\frac{\pi}{3}\right) = 5 \\ y &= r \sin \Theta = 10 \sin\left(\frac{\pi}{3}\right) = 5\sqrt{3} \\ z &= z = 7 \quad (\text{straight up the cylinder}) \end{aligned}$$



Cylindrical coordinates overlapping
Cartesian/Rectangular coordinates

Spherical Coordinate System

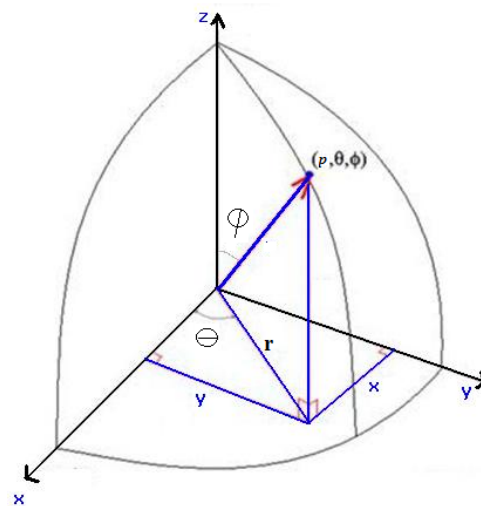
$$(p, \Theta, \Phi)$$

Converting rectangular coordinates (x, y, z) to spherical coordinates (p, Θ, Φ)

$$x = p \sin \Theta \cos \Phi \quad p^2 = x^2 + y^2 + z^2$$

$$y = p \sin \Theta \sin \Phi$$

$$z = p \cos \Theta$$



Converting spherical coordinates (p, Θ, Φ) to cylindrical coordinates (r, Θ, z)

$$r = p \sin \Theta \quad p^2 = r^2 + z^2$$

$$\Theta = \Theta$$

$$z = p \cos \Theta$$

Example: Convert $(1, -1, \sqrt{2})$ to spherical coordinates

$$p^2 = x^2 + y^2 + z^2$$

$$p = \sqrt{1^2 + (-1)^2 + (\sqrt{2})^2} = 2$$

$$z = p \cos \Theta$$

$$\sqrt{2} = 2 \cos \Theta$$

$$\cos \Theta = \frac{\sqrt{2}}{2}$$

$$\Theta = \frac{\pi}{4}$$

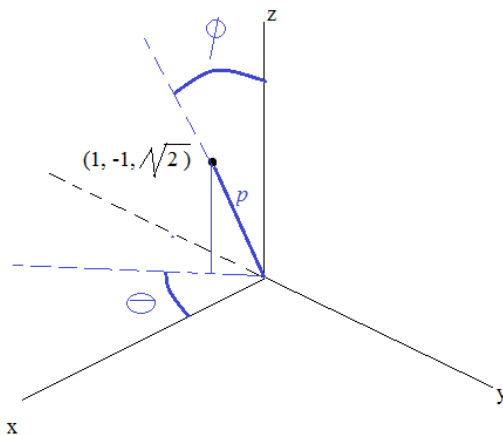
$$y = p \sin \Theta \sin \Phi$$

$$\sin \Theta = \frac{y}{p \sin \Phi}$$

$$\sin \Theta = \frac{-1}{2 \sin \frac{\pi}{4}}$$

$$\sin \Theta = \frac{-1}{\sqrt{2}}$$

$$\Theta = -\frac{\pi}{4}$$

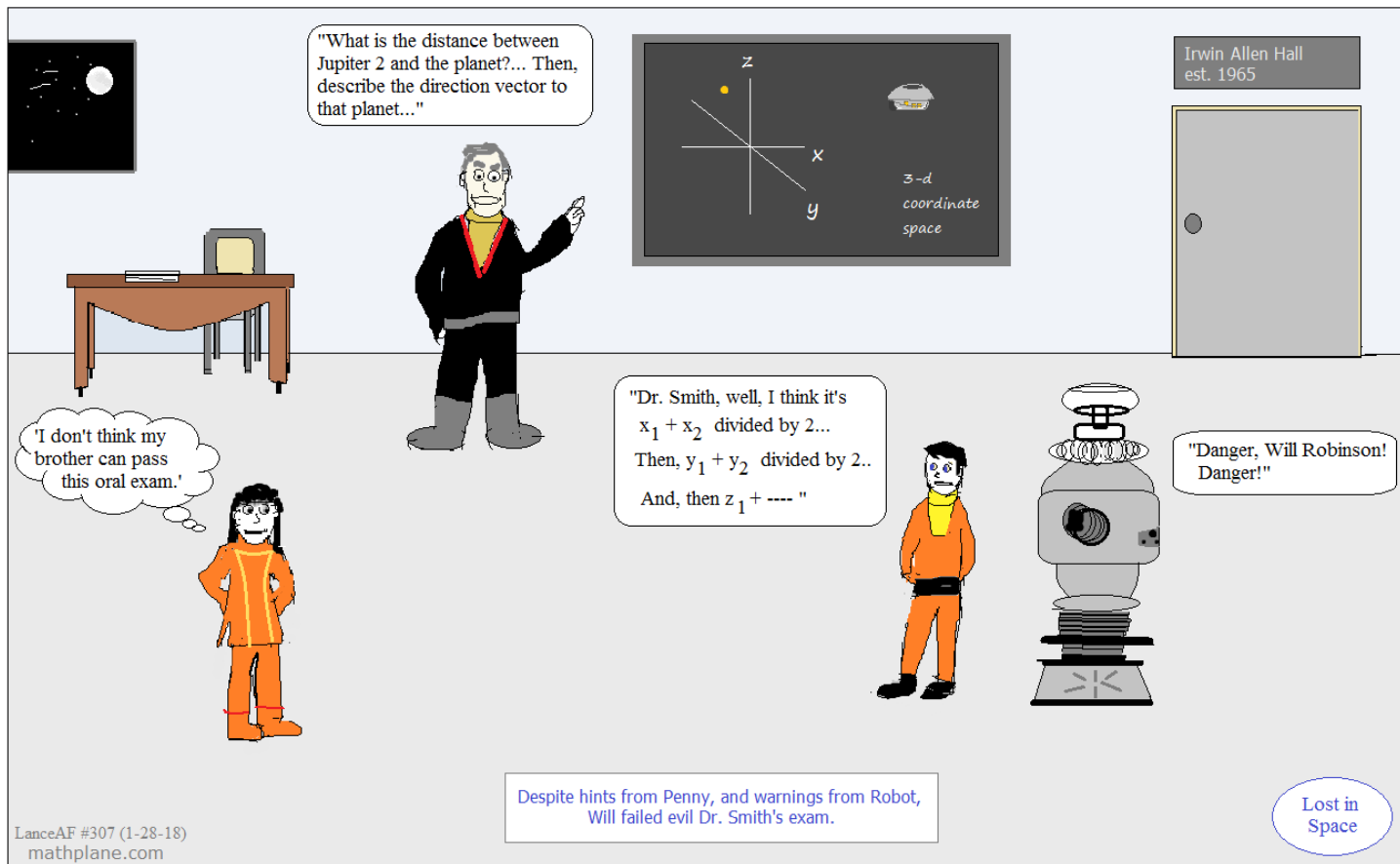


Φ shows the "swivel" along the yz-plane

Θ shows the "swivel" along the xy-plane

p shows the "radius" of the "sphere"

$$(2, -\frac{\pi}{4}, \frac{\pi}{4})$$

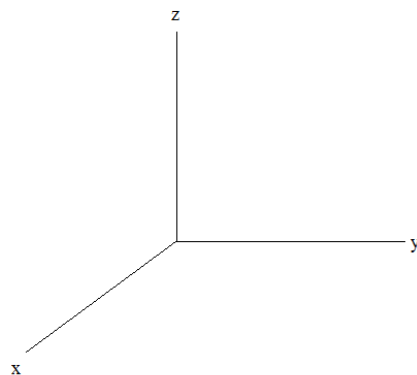


Exercises and Solutions→

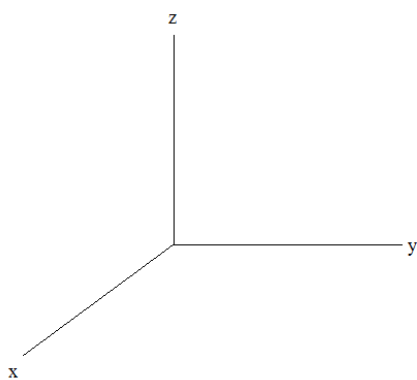
Sketch the following 3-dimensional figures....

3-Dimensional Sketches

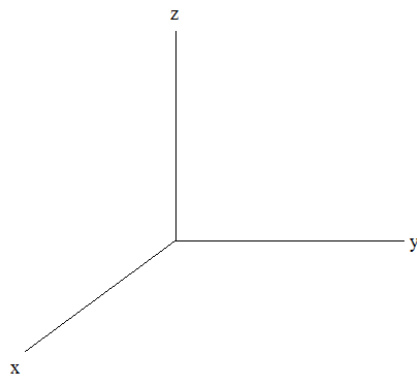
1) $4y^2 + 2z^2 = 16$ revolved around the z-axis...



2) $6y - y^2 = x^2$ revolved around the y-axis..

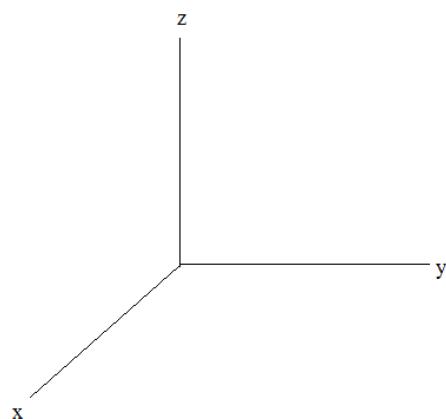


3) $x^2 - 2y^2 + 16y - 16 = 0$ revolved around y-axis

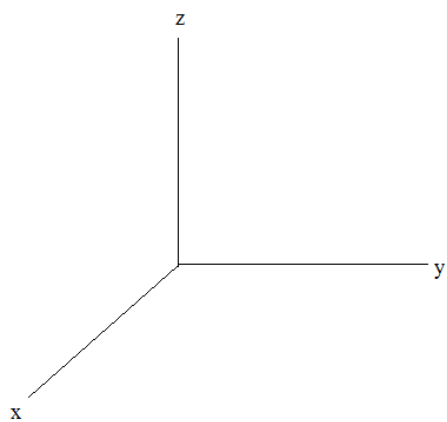


4) Sketch $2x^2 + 3y^2 + 3z^2 = 12$

3-Dimensional Sketches



5) Sketch $x^2 + y^2 - 5z^2 = 25$



Sketch the following 3-dimensional figures....

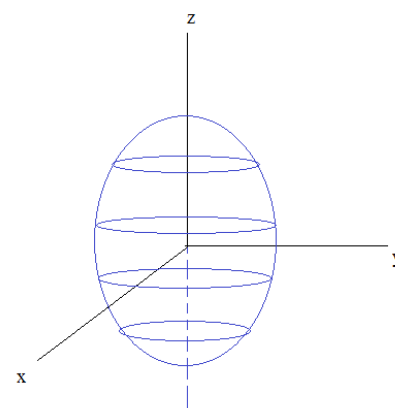
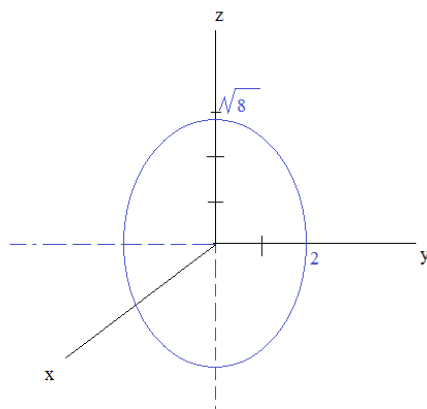
SOLUTIONS

3-Dimensional Sketches

- 1) $4y^2 + 2z^2 = 16$ revolved around the z-axis...

$$\frac{y^2}{4} + \frac{z^2}{8} = 1$$

an ellipse in the yz-plane...



- 2) $6y - y^2 = x^2$ revolved around the y-axis..

Step 1: Rewrite equation to find 2 dimensional figure...

$$x^2 + y^2 - 6y = 0$$

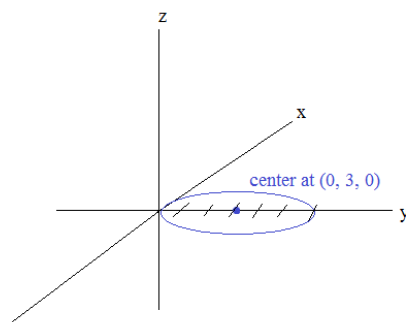
complete the square to put into standard form...

$$x^2 + y^2 - 6y + 9 = 0 + 9$$

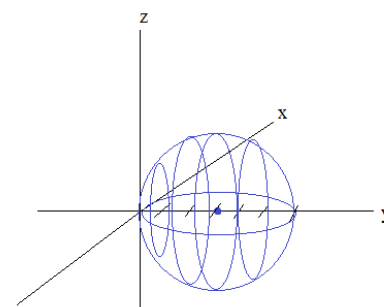
$$x^2 + (y-3)^2 = 9$$

center: (0, 3) in xy-plane or, (0, 3, 0)
radius: 3

Step 2: Sketch figure in xy-plane..



Step 3: Revolve around axis



3-d sphere

- 3) $x^2 - 2y^2 + 16y - 16 = 0$ revolved around y-axis

$$x^2 - 2(y^2 - 8y + \quad) = 16$$

(complete the square to put into standard form)

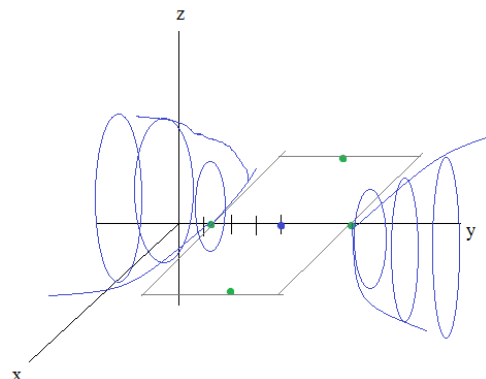
$$x^2 - 2(y^2 - 8y + 16) = 16 + (-32)$$

$$x^2 - 2(y-4)^2 = -16$$

$$\frac{(y-4)^2}{8} - \frac{x^2}{16} = 1$$

hyperbola in xy-plane center (0, 4)

or, (0, 4, 0)



4) Sketch $2x^2 + 3y^2 + 3z^2 = 12$

SOLUTIONS

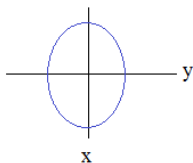
3-Dimensional Sketches

xy components

(when $z = 0$; xy-plane)

(ellipse)

$$\frac{x^2}{6} + \frac{y^2}{4} = 1$$

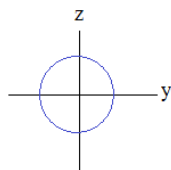


yz components

(when $x = 0$; yz-plane)

(circle)

$$y^2 + z^2 = 4$$

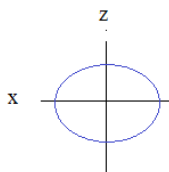


xz components

(when $y = 0$; xz-plane)

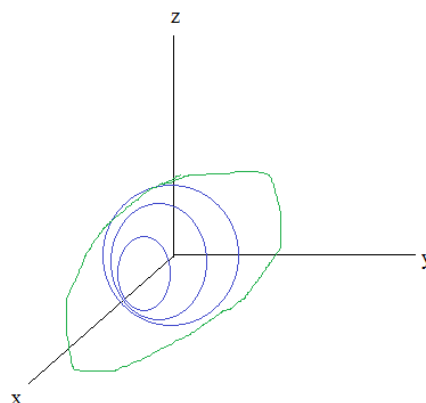
(ellipse)

$$\frac{x^2}{6} + \frac{z^2}{4} = 1$$



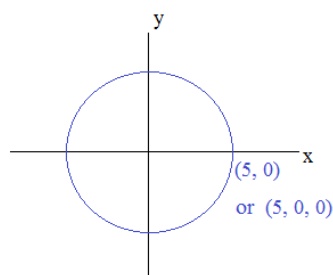
blended together it is a "football"
or, "spherical ellipsoid"

hollow



5) Sketch $x^2 + y^2 - 5z^2 = 25$ If we sketch the cross sections....

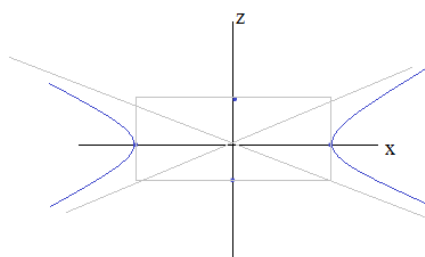
xy - plane (when $z = 0$)



$$x^2 + y^2 = 25$$

circle with radius 5,
centered at origin

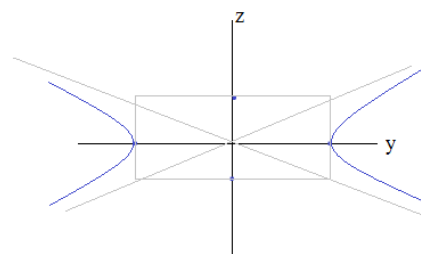
xz - plane (when $y = 0$)



$$\frac{x^2}{25} - \frac{z^2}{5} = 1$$

hyperbola centered at origin
and vertices at $(5, 0, 0)$
and $(-5, 0, 0)$

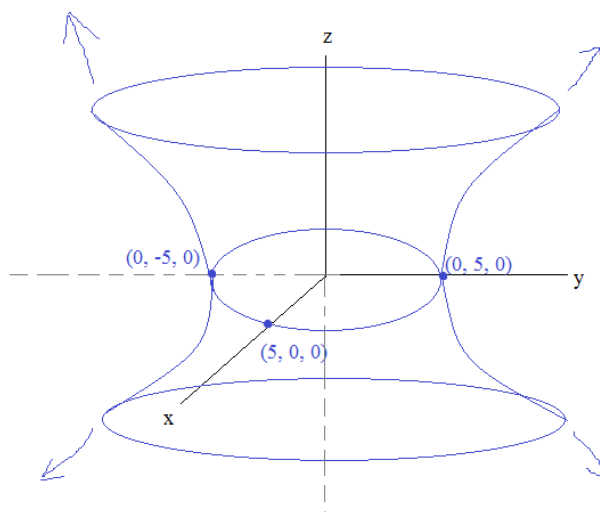
yz - plane (when $x = 0$)



$$\frac{y^2}{25} - \frac{z^2}{5} = 1$$

hyperbola centered at origin
and vertices at $(0, 5, 0)$
and $(0, -5, 0)$

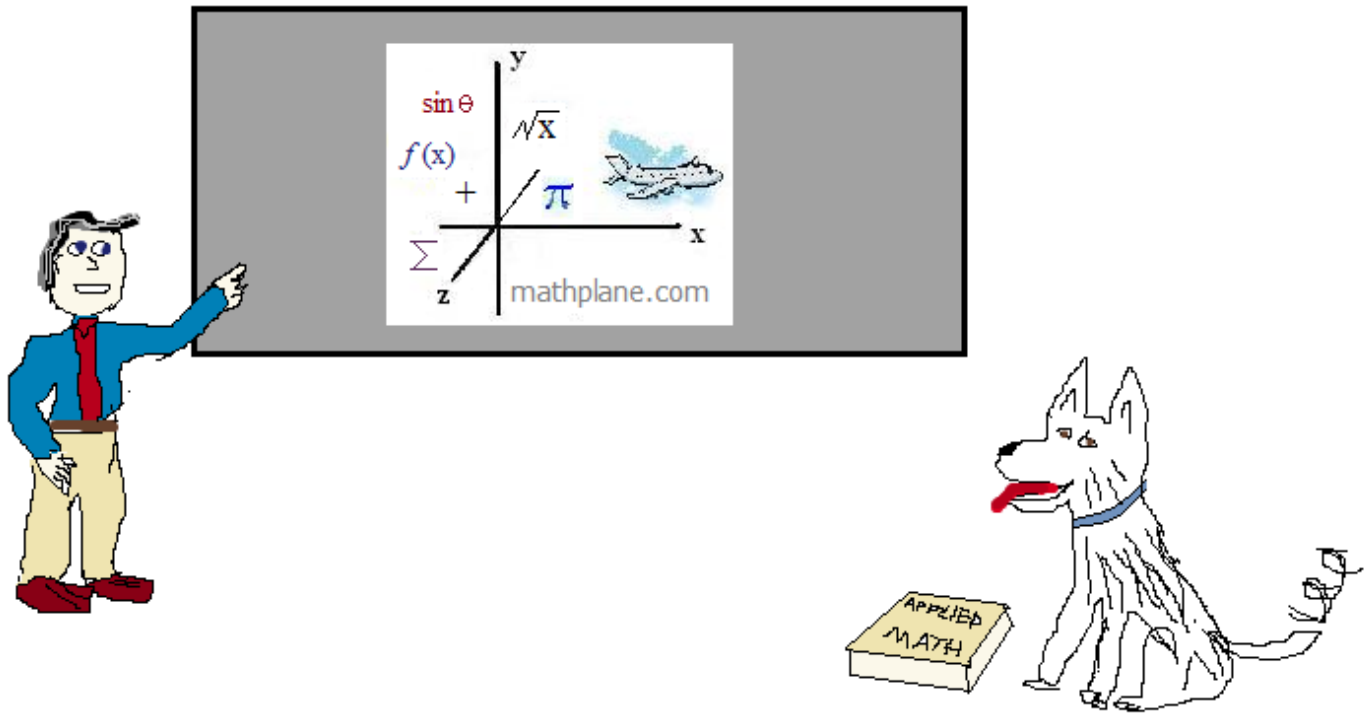
Then, combine them in the 3-d plane...



Thanks for visiting! (hope it helped)

If you have questions, suggestions, or requests, let us know.

Cheers.



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